Brownian motion and Stochastic Calculus
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Assignment 4—solutions

Exercise 1

Show the results stated in Remark 2.1.2 in the lecture notes.

For fixed $0 \leq s \leq t < +\infty$, arbitrary integer $n \in \mathbb{N}^\star$, and indices $0 =: s_0 < s_1 < \cdots < s_n =: s$, the σ-algebra $\sigma(X_0, X_{s_1}, \ldots, X_{s_n})$ coincides obviously with the σ-algebra $\sigma(X_0, X_{s_1} - X_0, \ldots, X_{s_n} - X_{s_{n-1}})$, and is $\mathbb{P}$-independent of $X_t - X_s$ by assumption. The union of all σ-algebrae of this form when n varies over $\mathbb{N}^\star$ constitutes a collection $\mathcal{C}$ of sets which is closed under finite intersection. Now the collection $\mathcal{D}$ of all sets in $\mathcal{F}_s^X$ which are $\mathbb{P}$-independent of $X_t - X_s$ is a Dynkin system containing $\mathcal{C}$. We conclude by Dynkin's π–λ-systems lemma, that $\mathcal{F}_s^X = \sigma(\mathcal{C})$ is contained in $\mathcal{D}$, which ends the proof.

Exercise 2

Prove Proposition 2.2.3.

Since all the Brownian motions are considered with respect to their natural filtrations, we can make implicitly use of Exercise 1 and check the independence property directly on the increments, when necessary.

(i) **The continuity of paths, as well as the $\mathbb{P}$-independence of the increments are immediate. As for the $\mathbb{P}$-distribution of the increments, it comes directly for the symmetry of Gaussian distributions, meaning that if $X \sim \mathcal{N}(\mu, \Sigma)$, then $AX \sim \mathcal{N}(A\mu, A\Sigma A^\top)$.**

(ii) **Similarly, the continuity of paths, as well as the $\mathbb{P}$-independence of the increments, are also immediate. Furthermore, for any $0 \leq s \leq t$**

$$X_t^c - X_s^c = c^{-1/2}\big(B_{ct} - B_{cs}\big) \sim \mathcal{N}(0, c^{-1}c(t-s) = t-s).$$

(iii) **Again, everything is rather straightforward as any increment of the process $(B_{t+s} - B_t)_{t\geq 0}$ is also an increment of B. The fact $(B_{t+s} - B_s)_{t\geq 0}$ is $\mathbb{P}$-independent of $(B_u)_{0\leq u\leq s}$ is also immediate from the fact that by definition of Brownian motion, for any $t \geq 0$, $B_{t+s} - B_s$ is $\mathbb{P}$-independent of $\mathcal{F}_s^B$.**

(iv) **This is the only non-trivial statement here, and we will use Proposition 2.2.2 in the Lecture Notes to obtain the result. It is clear that W is a Gaussian process under $\mathbb{P}$ (linear combinations of values of W are immediately linear combinations of values of B), with mean function 0. As for the covariance function, we have for any $(s,t) \in (0, +\infty)^2$**

$$K_W(s,t) = ts\mathbb{C}\mathrm{ov}\big[B_{1/t}, B_{1/s}\big] = ts\big(t^{-1} \wedge s^{-1}\big) = t \wedge s.$$

Similarly, if either s or t is equal to 0, since $W_0 = 0$, then $K_W(s,t) = 0$. The only remaining thing to check is that the paths of W are continuous. This is obvious on $(0, +\infty)$, but not so clear at 0. Notice that by continuity of W on $(0, +\infty)$ and of B on $[0, +\infty)$

$$\mathbb{P}\Big[\Big\{\lim_{t\downarrow 0} W_t = 0\Big\}\Big] = \mathbb{P}\left[\bigcap_{n\in\mathbb{N}^\star}\bigcup_{m\in\mathbb{N}^\star}\bigcap_{q\in\mathbb{Q}\cap(0,1/m]}\left\{\|W_q\| \leq \frac{1}{n}\right\}\right] = \mathbb{P}\left[\bigcap_{n\in\mathbb{N}^\star}\bigcup_{m\in\mathbb{N}^\star}\bigcap_{q\in\mathbb{Q}\cap(0,1/m]}\left\{\|B_q\| \leq \frac{1}{n}\right\}\right] = 1,$$

since the processes $(B_t)_{t>0}$ and $(W_t)_{t>0}$ are Gaussian processes under $\mathbb{P}$ with the same mean and covariance functions, so that they have the same distribution.

Exercise 3

Prove Proposition 2.2.4.

Before starting the proof, notice that for any map $f:[0,+\infty)\times\mathbb{R}\longrightarrow\mathbb{R}^d$ with at most exponential growth in the space variables, that is to say such that there exists some continuous map $C:[0,+\infty)\longrightarrow(0,+\infty)$ verifying

$$\big|f(t,x)\big|\leq C(t)\mathrm{e}^{C(t)\|x\|},\ (t,x)\in[0,+\infty)\times\mathbb{R}^d,$$

the properties of the Gaussian distribution imply that

$$\mathbb{E}^{\mathbb{P}}\big[\big|f(t,B_t)\big|\big]=\int_{\mathbb{R}^d}\big|f(t,x)\big|\phi_{0,t\mathrm{I}_d,d}(x)\mathrm{d}x\leq\frac{C(t)}{(2\pi t)^{d/2}}\int_{\mathbb{R}^d}\mathrm{e}^{C(t)\|x\|-\frac{1}{2t}\|x\|^2}\mathrm{d}x<+\infty.$$

This implies that all the functionals appearing in the statement are all $\mathbb{P}$-integrable. For further use, we also recall that if X is, under $\mathbb{P}$, a one-dimensional Gaussian random variable with mean 0 and variance σ^2, then for any integer p

$$\mathbb{E}^{\mathbb{P}}\big[X^{2p}\big]=\sigma^{2p}\frac{(2p)!}{2^p p!},\ \mathbb{E}^{\mathbb{P}}\big[X^{2p+1}\big]=0.$$

(i) **It suffices here to notice that by independence of the increments of B, and the fact that B is $\mathbb{F}$-adapted, we have for any $0\leq s\leq t$**

$$\mathbb{E}^{\mathbb{P}}[B_t|\mathcal{F}_s]=\mathbb{E}^{\mathbb{P}}[B_t-B_s+B_s|\mathcal{F}_s]=\mathbb{E}^{\mathbb{P}}[B_t-B_s|\mathcal{F}_s]+B_s=\mathbb{E}^{\mathbb{P}}[B_t-B_s]+B_s=B_s.$$

(ii) **We use the same trick as above and make increments of B appear. We have for any $0\leq s\leq t$**

$$\mathbb{E}^{\mathbb{P}}[\|B_t\|^2-dt|\mathcal{F}_s]=\mathbb{E}^{\mathbb{P}}\big[\|B_t-B_s\|^2\big|\mathcal{F}_s\big]+2B_s\cdot\mathbb{E}^{\mathbb{P}}\big[B_t-B_s|\mathcal{F}_s\big]+\|B_s\|^2-dt=d(t-s)+0+\|B_s\|^2-dt=\|B_s\|^2-ds.$$

(iii) **The reasoning is still the same and simply uses the classical formula for the Laplace transform of a Gaussian vector. We have for any $0\leq s\leq t$**

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\exp\big(\lambda\cdot B_t-t\|\lambda\|^2/2\big)\big)\big|\mathcal{F}_s\big]=\mathrm{e}^{\lambda\cdot B_s-t\|\lambda\|^2/2}\mathbb{E}^{\mathbb{P}}\big[\exp\big(\lambda\cdot(B_t-B_s)\big)\big|\mathcal{F}_s\big]&=\mathrm{e}^{\lambda\cdot B_s-t\|\lambda\|^2/2}\mathbb{E}^{\mathbb{P}}\big[\exp\big(\lambda\cdot(B_t-B_s)\big)\big]\\&=\exp\big(\lambda\cdot B_s-t\|\lambda\|^2/2+(t-s)\|\lambda\|^2/2\big)\\&=\exp\big(\lambda\cdot B_s-s\|\lambda\|^2/2\big).\end{aligned}$$

(iv) **For this last property, the intuition is still the same. We have for any $0\leq s\leq t$**

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[P_n(t,B_t)\big|\mathcal{F}_s\big]&=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k n!}{2^k k!(n-2k)!}t^k\mathbb{E}^{\mathbb{P}}\big[B_t^{n-2k}\big|\mathcal{F}_s\big]\\&=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k n!}{2^k k!(n-2k)!}t^k\bigg(\sum_{\ell=0}^{n-2k}\binom{n-2k}{\ell}B_s^{n-2k-\ell}\mathbb{E}^{\mathbb{P}}\big[(B_t-B_s)^\ell\big|\mathcal{F}_s\big]\bigg)\\&=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k n!}{2^k k!(n-2k)!}t^k\bigg(\sum_{\ell=0}^{n-2k}\binom{n-2k}{\ell}B_s^{n-2k-\ell}\mathbb{E}^{\mathbb{P}}\big[(B_t-B_s)^\ell\big]\bigg)\\&=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k n!}{2^k k!(n-2k)!}t^k\bigg(\sum_{\ell=0}^{\lfloor n/2\rfloor-k}\binom{n-2k}{2\ell}B_s^{n-2k-2\ell}\frac{(t-s)^\ell(2\ell)!}{2^\ell\ell!}\bigg)\\&=\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{\ell=k}^{\lfloor n/2\rfloor}\binom{\ell}{k}\frac{n!(-t)^k(t-s)^{\ell-k}}{2^\ell\ell!(n-2\ell)!}B_s^{n-2\ell}\\&=\sum_{\ell=0}^{\lfloor n/2\rfloor}\frac{n!}{2^\ell\ell!(n-2\ell)!}B_s^{n-2\ell}\bigg(\sum_{k=0}^{\ell}\binom{\ell}{k}(-t)^k(t-s)^{\ell-k}\bigg)\\&=\sum_{\ell=0}^{\lfloor n/2\rfloor}\frac{n!(-s)^\ell}{2^\ell\ell!(n-2\ell)!}B_s^{n-2\ell}=P_n(s,B_s).\end{aligned}$$

Exercise 4

A continuous time stochastic process is called a *Brownian Bridge* if it is a Gaussian process with mean 0 and covariance function $s(1-t)$, $s < t$. Let W be a Brownian motion and consider the process $X = (X_t)_{0 \leq t \leq 1}$ defined by $X_t =: W_t - tW_1$.

1) Show that X is a Brownian Bridge, and that X does **not** have independent increments.

2) Show that if $(X_n)_{n \in \mathbb{N}}$ is a Gaussian process indexed by $\mathbb{N}$ and converges in probability to a random variable X as n goes to infinity, then it converges also in $\mathbb{L}^2(\mathbb{R}, \mathcal{F}, \mathbb{P})$ to X.

1) **We need to show that for any $n \in \mathbb{N}^\star$ and any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1$, the random vector $(X_{t_1}, \dots, X_{t_n})$ is a Gaussian vector. It thus actually suffices to show that $(X_{t_1}, \dots, X_{t_n})$ is the image of a linear transformation of another Gaussian vector. From the lecture notes, we know that the Brownian motion W is a Gaussian process. We distinguish between two cases**

case 1: $t_n < 1$

In this case, the vector $(X_{t_1}, \dots, X_{t_n})$ is the image of the Gaussian vector $(W_{t_1}, \dots, W_{t_n}, W_1)$ under the linear map $A : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ defined by

$$A_{ij} := \begin{cases} 1, \ i = j \in \{1, \dots, n\}, \\ -t_i, \ j = n+1, \ i \in \{1, \dots, n\} \\ 0, \ \text{else.} \end{cases}$$

case 2: $t_n = 1$

In that case, the vector $(X_{t_1}, \dots, X_{t_n})$ is the image of the Gaussian vector $(W_{t_1}, \dots, W_{t_{n-1}}, W_1)$ under the linear map $B : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$B_{ij} = \begin{cases} 1, \ i = j \in \{1, \dots, n-1\}, \\ -t_i, \ j = n, \ i \in \{1, \dots, n-1\} \\ 0, \ \textbf{else.} \end{cases}$$

In both cases, $(X_{t_1}, \dots, X_{t_n})$ is the image of a linear transformation of a Gaussian vector, hence we are done.

For any $t \in [0, 1]$ we have

$$\mathbb{E}^{\mathbb{P}}[X_t] = \mathbb{E}^{\mathbb{P}}[W_t - tW_1] = 0.$$

For any $0 \leq s, t \leq 1$, using that $\mathbb{C}\text{ov}(W_t, W_s) = t \wedge s$, we have

$$\mathbb{C}\text{ov}(X_t, X_s) = \mathbb{C}\text{ov}(W_t, W_s) - s\mathbb{C}\text{ov}(W_t, W_1) - t\mathbb{C}\text{ov}(W_1, W_s) + ts\mathbb{C}\text{ov}(W_1, W_1) = t \wedge s - ts.$$

Take any $t \in (0, 1)$. We show that the increments $X_1 - X_t$, $X_t - X_0$ are correlated. In the same way as above we obtain that

$$\mathbb{C}\text{ov}(X_1 - X_t, X_t - X_0) = \mathbb{C}\text{ov}(-W_t + tW_1, W_t - tW_1) = t(t-1) \neq 0.$$

2) **It is known from Probability theory course that convergence in probability implies convergence in distribution. Hence,**

$$\varphi_n(t) := \mathbb{E}e^{itX_n} \longrightarrow \varphi(t) := \mathbb{E}e^{itX}, \quad t \in \mathbb{R}. \tag{0.1}$$

Since X_n are Gaussian, we have

$$\varphi_n(t) = \exp\left(it\mu_n - \frac{1}{2}t^2\sigma_n^2\right), \quad t \in \mathbb{R},$$

where $\mu_n = \mathbb{E}X_n$ **and** $\sigma_n = \mathrm{var}(X_n)$. **It is also clear from** (0.1) **that, necessarily,**

$$|\varphi_n(t)| \longrightarrow |\varphi(t)|, \quad t \in \mathbb{R}.$$

Because $|\varphi_n(t)| = \exp(-\frac{1}{2}t^2\sigma_n^2)$, **this is possible only if the sequence** $(\sigma_n^2)_{n=1}^{\infty}$ **admits a limit, which we denote by** σ^2. **Plugging this into** (0.1) **yields that the sequence** $(\mu_n)_{n=1}^{\infty}$ **also necessarily admits a limit, which we denote by** μ. **Hence,**

$$\varphi(t) = \exp\left(it\mu - \frac{1}{2}t^2\sigma^2\right), \quad t \in \mathbb{R}.$$

That is to say, $X \sim \mathcal{N}(\mu, \sigma^2)$.

It is also clear that

$$\sup_{n\in\mathbb{N}} \mathbb{E}(X_n)^4 = \sup_{n\in\mathbb{N}} \left(\mu_n^4 + 6\mu_n^2\sigma_n^2 + 3\sigma_n^4\right) < \infty.$$

This yields that the sequence (X_n^2) **is uniformly integrable, which together with convergence in probability gives**

$$X_n \longrightarrow X \quad \textbf{in } L^2(\mathbb{R}, \mathcal{F}, \mathbb{P}).$$